A new algebra which transmutes to the braided algebra

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Abstract

We find a new braided Hopf structure for the algebra satisfied by the entries of the braided matrix $BSL_q(2)$. A new nonbraided algebra whose coalgebra structure is the same as the braided one is found to be a two parameter deformed algebra. It is found that this algebra is not a comodule algebra under adjoint coaction. However, it is shown that for a certain value of one of the deformation parameters the braided algebra becomes a comodule algebra under the coaction of this nonbraided algebra. The transmutation of the nonbraided algebra to the braided one is constructed explicitly.

1 Introduction

The covariances of algebraic structures are of central importance for the physical and mathematical theories constructed by using these algebraic structures. The investigation of quantum group covariant structures gives rise to the introduction of braided groups (a self contained review can be found in [1]) which are obtained from quantum groups by a covariantization process called transmutation . Quantum groups of function algebra type are not invariant under their own transformation, i.e., the quantum algebra $A(R)$ is not a comodule algebra under the adjoint coaction. However, it is possible to obtain an $A(R)$-comodule algebra called braided algebra $B(R)$ obtained by covariantizing the algebra of the coacted copy keeping the coalgebra unchanged. This covariantization process deforms the notion of tensor product and leads to the generalization of the usual Hopf algebra axioms called braided Hopf algebra axioms [2]. Thus the braided Hopf algebras can be used to generalize the supersymmetric structures via the generalization of super tensor product and super Hopf algebras [3] or to introduce oscillators with braid statistics [4].

In this work, we investigate the covariance properties of the algebra satisfied by the entries of the braided matrix $BSL_q(2)$. In other words, we investigate if there is any Hopf algebra other the quantum algebra the coaction of which makes the braided algebra a comodule algebra. Since the coacting and the coacted algebras have the same coalgebra structure in transmutation theory, we first investigate the general braided Hopf algebra structure. We find that there is one more braided Hopf

algebra other the one given in the literature. We also find that the nonbraided Hopf algebra whose coproduct is the same as the braided one is a two parameter deformed Hopf algebra. This algebra, like the quantum algebra, is not a comodule algebra under the adjoint coaction. For a certain value of one of the deformation parameters it turns out that the braided algebra becomes a comodule algebra under the coaction of this one parameter nonbraided algebra. We explicitly construct the transmutation of the noncovariant (nonbraided) algebra to the covariant (braided) one.

2 Prelimineries and Review

The right coaction of a Hopf algebra A on H is a linear map $\beta : H \to H \otimes A$, i.e.,

$$\beta(h) = \Sigma h^{(i)} \otimes a^{(i)}, \quad h^{(i)} \in H, \quad a^{(i)} \in A \tag{1}$$

satisfying

$$(\beta \otimes id) \circ \beta = (id \otimes \Delta) \circ \beta \quad (id \otimes \epsilon) \circ \beta = id. \tag{2}$$

The algebra H is a right A-comodule algebra if the map β is an algebra homomorphism such that

$$\beta(h \cdot g) = \beta(h) \cdot \beta(g) \quad \forall h, g \in H. \tag{3}$$

The consistency of the algebra homomorphism requires that $\beta(1_H) = 1_H \otimes 1_A$. The right adjoint coaction of a Hopf algebra on itself is defined by

$$\beta(h) = \Sigma h_{(2)} \otimes S(h_{(1)}) \cdot h_{(3)} \tag{4}$$

where $h_{(1)}, h_{(2)}, h_{(3)}$ are given by

$$\Delta^2(h) = \Sigma h_{(1)} \otimes h_{(2)} \otimes h_{(3)}. \tag{5}$$

The quantum algebra $(A(R))$ of the quantum matrix $SL_q(2)$ is generated by a, b, c, d and 1 satisfying the relations

$$\begin{aligned} a \cdot b &= q^{-1} b \cdot a, \\ a \cdot c &= q^{-1} c \cdot a, \\ b \cdot d &= q^{-1} d \cdot b, \\ c \cdot d &= q^{-1} d \cdot c, \\ b \cdot c &= c \cdot b, \end{aligned} \tag{6}$$

$$\begin{aligned} a \cdot d - d \cdot a &= (q^{-1} - q) b \cdot c \\ a \cdot d - q^{-1} b \cdot c &= \; = 1 \end{aligned}$$

the Hopf structure of which is given by the coproducts

$$\begin{aligned} \Delta(a) &= a \otimes a + b \otimes c, \\ \Delta(b) &= a \otimes b + b \otimes d, \\ \Delta(c) &= c \otimes a + d \otimes c, \\ \Delta(d) &= c \otimes b + d \otimes d \end{aligned} \tag{7}$$

the counits

$$\epsilon(a) = 1, \quad \epsilon(b) = 0, \quad \epsilon(c) = 0, \quad \epsilon(d) = 1, \tag{8}$$

and by the antipodes

$$S(a) = d, \quad S(b) = -qb, \quad S(c) = -q^{-1}c, \quad S(d) = a. \tag{9}$$

The $*$-algebra structure with real q is given by

$$a^* = d, \quad b^* = -qb, \quad c^* = -q^{-1}c, \quad d^* = a. \tag{10}$$

The adjoint coaction of $A(R)$ on itself is then calculated to give

$$\begin{aligned} \beta(a) &= a \otimes d \cdot a + b \otimes d \cdot c + c \otimes (-qb \cdot a) + d \otimes (-qb \cdot c) \\ \beta(b) &= a \otimes d \cdot b + b \otimes d \cdot d + c \otimes (-qb \cdot c) + d \otimes (-qb \cdot d) \\ \beta(c) &= a \otimes (-q^{-1}c \cdot a) + b \otimes (-q^{-1}c \cdot c) + c \otimes a \cdot a + d \otimes a \cdot c \\ \beta(d) &= a \otimes (-q^{-1}c \cdot b) + b \otimes (-q^{-1}c \cdot d) + c \otimes a \cdot b + d \otimes a \cdot d. \end{aligned} \tag{11}$$

It can easily be seen that these mappings do not define an algebra homomorphism. Hence $A(R)$ itself is not an $A(R)$-comodule algebra. However, the multiplication $(\cdot)$ of the coacted copy (first elements in the tensor product) is replaced by a multiplication $(\underline{\cdot})$ such that

$$\begin{aligned} a \underline{\cdot} a &= a \cdot a \\ a \underline{\cdot} b &= a \cdot b \\ a \underline{\cdot} c &= qc \cdot a \\ a \underline{\cdot} d &= a \cdot d + (q - q^{-1}) c \cdot b \\ b \underline{\cdot} a &= q^2 a \cdot b \end{aligned}$$

$$
\begin{aligned}
b\underline{\cdot}b &= qb\cdot b \\
b\underline{\cdot}c &= q^{-1}b\cdot c+(1-q^{-2})(d-a)\cdot a \\
b\underline{\cdot}d &= qb\cdot d-(1-q^{-2})a\cdot b \\
c\underline{\cdot}a &= q^{-1}c\cdot a \\
c\underline{\cdot}b &= q^{-1}c\cdot b \\
c\underline{\cdot}c &= qc\cdot c \\
c\underline{\cdot}d &= qc\cdot d \\
d\underline{\cdot}a &= d\cdot a \\
d\underline{\cdot}b &= d\cdot b \\
d\underline{\cdot}c &= d\cdot c-q^{-1}(1-q^{-2})c\cdot a \\
d\underline{\cdot}d &= d\cdot d-q^{-1}(1-q^{-2})c\cdot b.
\end{aligned}
\tag{12}
$$

Then the algebra $(B(R))$ satisfied by a, b, c, d and 1 with this new multiplication $(\underline{\cdot})$

$$
\begin{aligned}
b\underline{\cdot}a &= q^{2}a\underline{\cdot}b, \\
c\underline{\cdot}a &= q^{-2}a\underline{\cdot}c, \\
a\underline{\cdot}d &= d\underline{\cdot}a, \\
b\underline{\cdot}c &= c\underline{\cdot}b \;+\; (1-q^{-2})a\underline{\cdot}(d-a), \\
d\underline{\cdot}b &= b\underline{\cdot}d \;+\; (1-q^{-2})a\underline{\cdot}b, \\
c\underline{\cdot}d &= d\underline{\cdot}c \;+\; (1-q^{-2})c\underline{\cdot}a, \\
a\underline{\cdot}d-q^{2}c\underline{\cdot}b &= 1.
\end{aligned}
\tag{13}
$$

is an $A(R)$-comodule algebra under the coaction (11). This covariantization process is called transmutation. The transformation under a noncommutative algebra ($A(R)$ for instance) deforms the notion of tensor product because the transformed algebras are no longer independent. The deformed tensor product is called the braided tensor product and denoted by $\underline{\otimes}$. The coalgebra of the transmuted algebra $B(R)$ is of the same form as the original algebra $A(R)$, i.e.,

$$
\begin{aligned}
\underline{\Delta}(a) &= a\underline{\otimes}a+b\underline{\otimes}c, \\
\underline{\Delta}(b) &= a\underline{\otimes}b+b\underline{\otimes}d, \\
\underline{\Delta}(c) &= c\underline{\otimes}a+d\underline{\otimes}c, \\
\underline{\Delta}(d) &= c\underline{\otimes}b+d\underline{\otimes}d \\
\underline{\epsilon}(a) &= \underline{\epsilon}(d)=1,\ \underline{\epsilon}(b)=\underline{\epsilon}(c)=0.
\end{aligned}
\tag{14}
$$

The antipodes of the generators of $B(R)$ are given by

$$\underline{S}(a)=q^2d+(1-q^2)a,\quad \underline{S}(b)=-q^2b,\quad \underline{S}(c)=-q^2c,\quad \underline{S}(d)=a \tag{15}$$

The coproducts define an algebra homomorphism in the braided tensor product space with the braided tensor product algebra such that

$$(x\underline{\otimes}y)\underline{\cdot}(w\underline{\otimes}z)=x\psi(y\underline{\otimes}z)d \quad x,y,w,z\in B. \tag{16}$$

where ψ is a generalization of the permutation map in boson algebras. The braiding relations define a nontrivial statistics between copies of algebras. The braiding relations for the braided algebra (13) with the coalgebra (14) are given by

$$\begin{aligned}
\psi(a\underline{\otimes}a) &= a\underline{\otimes}a+(1-q^2)b\underline{\otimes}c\\
\psi(a\underline{\otimes}b) &= b\underline{\otimes}a\\
\psi(a\underline{\otimes}c) &= c\underline{\otimes}a+(1-q^2)(d-a)\underline{\otimes}c\\
\psi(a\underline{\otimes}d) &= d\underline{\otimes}a+(1-q^{-2})b\underline{\otimes}c\\
\psi(b\underline{\otimes}a) &= a\underline{\otimes}b+(1-q^2)b\underline{\otimes}(d-a)\\
\psi(b\underline{\otimes}b) &= q^2b\underline{\otimes}b\\
\psi(b\underline{\otimes}c) &= q^{-2}c\underline{\otimes}b+(1+q^2)(1-q^2)^2b\underline{\otimes}c-(1-q^{-2})(d-a)\underline{\otimes}(d-a)\\
\psi(b\underline{\otimes}d) &= d\underline{\otimes}b+(1-q^{-2})b\underline{\otimes}(d-a)\\
\psi(c\underline{\otimes}a) &= a\underline{\otimes}c\\
\psi(c\underline{\otimes}b) &= q^{-2}b\underline{\otimes}c\\
\psi(c\underline{\otimes}c) &= q^2c\underline{\otimes}c\\
\psi(c\underline{\otimes}d) &= d\underline{\otimes}c\\
\psi(d\underline{\otimes}a) &= a\underline{\otimes}d+(1-q^{-2})b\underline{\otimes}c\\
\psi(d\underline{\otimes}b) &= b\underline{\otimes}d\\
\psi(d\underline{\otimes}c) &= c\underline{\otimes}d+(1-q^{-2})(d-a)\underline{\otimes}c\\
\psi(d\underline{\otimes}d) &= d\underline{\otimes}d-q^{-2}(1-q^{-2})b\underline{\otimes}c.
\end{aligned} \tag{17}$$

(18)

The central element $q^{-1}a+qd$ in quantum algebra (which is the quantum trace in the matrix algebra) is not only central in the braided algebra but also bosonic in the sense that

$$\psi((q^{-1}a+qd)\underline{\otimes}x)=x\underline{\otimes}(q^{-1}a+qd),\quad \psi(x\underline{\otimes}(q^{-1}a+qd))=(q^{-1}a+qd)\underline{\otimes}x\ \ \forall x\in B(R). \tag{19}$$

The algebras in the braided category satisfy the braided Hopf algebra axioms [2]

$$
\begin{aligned}
m \circ (id \otimes m) &= m \circ (m \otimes id) \\
m \circ (id \otimes \eta) &= m \circ (\eta \otimes id) = id \\
(id \otimes \Delta) \circ \Delta &= (\Delta \otimes id) \circ \Delta \\
(\epsilon \otimes id) \circ \Delta &= (id \otimes \epsilon) \circ \Delta = id \\
m \circ (id \otimes S) \circ \Delta &= m \circ (S \otimes id) \circ \Delta = \eta \circ \epsilon \\
\psi \circ (m \otimes id) &= (id \otimes m) \circ (\psi \otimes id) \circ (id \otimes \psi) \\
\psi \circ (id \otimes m) &= (m \otimes id) \circ (id \otimes \psi) \circ (\psi \otimes id) \\
(id \otimes \Delta) \circ \psi &= (\psi \otimes id) \circ (id \otimes \psi) \circ (\Delta \otimes id) \\
(\Delta \otimes id) \circ \psi &= (id \otimes \psi)(\psi \otimes id) \circ (id \otimes \Delta) \\
\Delta \circ m &= (m \otimes m)(id \otimes \psi \otimes id) \circ (\Delta \otimes \Delta) \\
S \circ m &= m \circ \psi \circ (S \otimes S) \\
\Delta \circ S &= (S \otimes S) \circ \psi \circ \Delta \\
\epsilon \circ m &= \epsilon \otimes \epsilon \\
(\psi \otimes id) \circ (id \otimes \psi) \circ (\psi \otimes id) &= (id \otimes \psi) \circ (\psi \otimes id) \circ (id \otimes \psi)
\end{aligned}
\tag{20}
$$

and the $\underline{*}$ algebras in the same category with real deformation parameter also satisfy [5]

$$
\begin{aligned}
\Delta \circ * &= \pi \circ (* \otimes *) \circ \Delta \\
S \circ * &= * \circ S \\
(x \otimes y)^* &= y^* \otimes x^*, \ \forall x, y \in B
\end{aligned}
\tag{21}
$$

where we omit the underlining for the sake of clarity and all mappings are in the braided sense in the axioms (20)-(21). Note that in the $\psi \to \pi$ limit the axioms (20) reduce to the usual Hopf algebra axioms. It is also possible to define more general Hopf algebras where the counit map is no longer an algebra homomorphism [6].

The involutions for (13)

$$
a^{\underline{*}} = a, \quad b^{\underline{*}} = c, \quad c^{\underline{*}} = d, \quad d^{\underline{*}} = d \tag{22}
$$

complete the braided Hopf $\underline{*}$-algebra structure.

3 A New Braided Hopf Algebra Solution

Expressing the braided algebra in terms of a central and bosonic element

$$p \equiv q^{-2}a + d \tag{23}$$

and three other generators a, b, c makes a lot simplification in the calculations. The algebra (13) can equivalently be expressed as

$$\begin{aligned}
b\underline{\cdot}a &= q^2 a\underline{\cdot}b \\
a\underline{\cdot}c &= q^2 c\underline{\cdot}a \\
b\underline{\cdot}c &= c\underline{\cdot}b - (1-q^{-4})a^2 + (1-q^{-2})p\underline{\cdot}a \\
q^{-2}a\underline{\cdot}a + a\underline{\cdot}p - q^2 c\underline{\cdot}b &= 1
\end{aligned} \tag{24}$$

The $\underline{*}$-structure (22) implies

$$a^{\underline{*}} = a, \quad b^{\underline{*}} = c, \quad c^{\underline{*}} = b, \quad p^{\underline{*}} = p. \tag{25}$$

We write the general forms of the coproducts

$$\begin{aligned}
\underline{\Delta}(a) &= A_1 a\underline{\otimes}a + A_2 b\underline{\otimes}c + A_3 c\underline{\otimes}b + A_4 p\underline{\otimes}a + A_5 a\underline{\otimes}p + A_6 1\underline{\otimes}a \\
&\quad + A_7 a\underline{\otimes}1 + A_8 1\underline{\otimes}1 + A_9 p\underline{\otimes}p + A_{10} 1\underline{\otimes}p + A_{11} p\underline{\otimes}1, \\
\underline{\Delta}(b) &= B_1 a\underline{\otimes}b + B_2 b\underline{\otimes}a + B_3 b\underline{\otimes}p + B_4 p\underline{\otimes}b + B_5 1\underline{\otimes}b + B_6 b\underline{\otimes}1, \\
\underline{\Delta}(c) &= B_1 c\underline{\otimes}a + B_2 a\underline{\otimes}c + B_3 p\underline{\otimes}c + B_4 c\underline{\otimes}p + B_5 c\underline{\otimes}1 + B_6 1\underline{\otimes}c, \\
\underline{\Delta}(p) &= C_1 a\underline{\otimes}a + C_2 b\underline{\otimes}c + C_3 c\underline{\otimes}b + C_4 p\underline{\otimes}a + C_5 a\underline{\otimes}p \\
&\quad + C_6 1\underline{\otimes}a + C_7 a\underline{\otimes}1 + C_8 1\underline{\otimes}1 + C_9 p\underline{\otimes}p + C_{10} 1\underline{\otimes}p + C_{11} p\underline{\otimes}1
\end{aligned} \tag{26}$$

the counits

$$\begin{aligned}
\underline{\epsilon}(a) &= e_1, \\
\underline{\epsilon}(b) &= \underline{\epsilon}(c) = e_2, \\
\underline{\epsilon}(p) &= e_3
\end{aligned} \tag{27}$$

the antipodes

$$\begin{aligned}
\underline{S}(a) &= k_1 a + k_2 b + k_3 c + k_4 p + k_5, \\
\underline{S}(b) &= m_1 a + m_2 b + m_3 c + m_4 p + m_5, \\
\underline{S}(c) &= m_1 a + m_2 c + m_3 b + m_4 p + m_5, \\
\underline{S}(p) &= n_1 a + n_2 b + n_3 c + n_4 p + n_5
\end{aligned} \tag{28}$$

the braidings

$$
\begin{aligned}
\psi(a\underline{\otimes}a) &= g_1 a\underline{\otimes}a + g_2 b\underline{\otimes}c + g_3 c\underline{\otimes}b + g_4 p\underline{\otimes}a + g_5 a\underline{\otimes}p \\
&\quad + g_6 1\underline{\otimes}a + g_7 a\underline{\otimes}1 + g_8 1\underline{\otimes}1 + g_9 p\underline{\otimes}p + g_{10} 1\underline{\otimes}p + g_{11} p\underline{\otimes}1, \\
\psi(a\underline{\otimes}b) &= d_1 b\underline{\otimes}a + d_2 a\underline{\otimes}b + d_3 p\underline{\otimes}b + d_4 b\underline{\otimes}p + d_5 1\underline{\otimes}b + d_6 b\underline{\otimes}1, \\
\psi(a\underline{\otimes}c) &= f_1 c\underline{\otimes}a + f_2 a\underline{\otimes}c + f_3 p\underline{\otimes}c + f_4 c\underline{\otimes}p + f_5 1\underline{\otimes}c + f_6 c\underline{\otimes}1, \\
\psi(b\underline{\otimes}b) &= z_1 b\underline{\otimes}b, \\
\psi(c\underline{\otimes}b) &= c_1 a\underline{\otimes}a + c_2 b\underline{\otimes}c + c_3 c\underline{\otimes}b + c_4 p\underline{\otimes}a + c_5 a\underline{\otimes}p \\
&\quad + c_6 1\underline{\otimes}a + c_7 a\underline{\otimes}1 + c_8 1\underline{\otimes}1 + c_9 p\underline{\otimes}p + c_{10} 1\underline{\otimes}p + c_{11} p\underline{\otimes}1, \\
\psi(b\underline{\otimes}c) &= a_1 a\underline{\otimes}a + a_2 b\underline{\otimes}c + a_3 c\underline{\otimes}b + a_4 p\underline{\otimes}a + a_5 a\underline{\otimes}p \\
&\quad + a_6 1\underline{\otimes}a + a_7 a\underline{\otimes}1 + a_8 1\underline{\otimes}1 + a_9 p\underline{\otimes}p + a_{10} 1\underline{\otimes}p + a_{11} p\underline{\otimes}1
\end{aligned}
\tag{29}
$$

and their $\underline{*}$-involutions to find the solutions for the braided Hopf algebra structure. The symbols with a subscript are the parameters to be determined. For the bosonic trace , the braiding of the central element p is trivial. We substitute these general forms into the braided Hopf algebra axioms (20) and solve the equations by using the computer programming Maple V. We find that that there are only two solutions.

For the first solution the coproducts

$$
\begin{aligned}
\underline{\Delta}(a) &= a\underline{\otimes}a + b\underline{\otimes}c \\
\underline{\Delta}(b) &= a\underline{\otimes}b - q^{-2} b\underline{\otimes}a + b\underline{\otimes}p \\
\underline{\Delta}(c) &= c\underline{\otimes}a - q^{-2} a\underline{\otimes}c + p\underline{\otimes}c \\
\underline{\Delta}(p) &= (q^{-2}+q^{-4}) a\underline{\otimes}a + q^{-2} b\underline{\otimes}c + c\underline{\otimes}b - q^{-2} p\underline{\otimes}a - q^{-2} a\underline{\otimes}p + p\underline{\otimes}p
\end{aligned}
\tag{30}
$$

the counits

$$
\begin{aligned}
\underline{\epsilon}(a) &= 1, \\
\underline{\epsilon}(b) &= 0 \\
\underline{\epsilon}(c) &= 0, \\
\underline{\epsilon}(p) &= 1 + q^{-2}
\end{aligned}
\tag{31}
$$

the antipodes

$$
\underline{S}(a) = q^2(p-a),
$$

$$
\begin{aligned}
\underline{S}(b) &= -q^2 b, \\
\underline{S}(c) &= -q^2 c, \\
\underline{S}(p) &= p
\end{aligned}
\tag{32}
$$

together with the braidings

$$
\begin{aligned}
\psi(a\underline{\otimes}a) &= a\underline{\otimes}a + (1-q^2)b\underline{\otimes}c, \\
\psi(a\underline{\otimes}b) &= b\underline{\otimes}a, \\
\psi(a\underline{\otimes}c) &= c\underline{\otimes}a + (q^2-q^{-2})a\underline{\otimes}c + (1-q^2)p\underline{\otimes}c, \\
\psi(b\underline{\otimes}b) &= q^2 b\underline{\otimes}b, \\
\psi(c\underline{\otimes}b) &= q^{-2} b\underline{\otimes}c, \\
\psi(b\underline{\otimes}c) &= (-1-q^{-2}+q^{-4}+q^{-6})a\underline{\otimes}a + (q^2-1-q^{-2}+q^{-4})b\underline{\otimes}c \\
&\quad +q^{-2}c\underline{\otimes}b + (1-q^{-4})a\underline{\otimes}p + (1-q^{-4})p\underline{\otimes}a + (q^{-2}-1)p\underline{\otimes}p
\end{aligned}
\tag{33}
$$

and their $\underline{*}$-involutions define a braided Hopf $\underline{*}$-algebra. For the second solution the coproducts

$$
\begin{aligned}
\underline{\Delta}(a) &= a\underline{\otimes}a + q^4 c\underline{\otimes}b, \\
\underline{\Delta}(b) &= -q^2 a\underline{\otimes}b + b\underline{\otimes}a + q^2 p\underline{\otimes}b, \\
\underline{\Delta}(c) &= -q^2 c\underline{\otimes}a + a\underline{\otimes}c + q^2 c\underline{\otimes}p, \\
\underline{\Delta}(p) &= (1+q^2)a\underline{\otimes}a + q^2 b\underline{\otimes}c + q^4 c\underline{\otimes}b - q^2 p\underline{\otimes}a - q^2 a\underline{\otimes}p + q^2 p\underline{\otimes}p
\end{aligned}
\tag{34}
$$

the counits

$$
\begin{aligned}
\underline{\epsilon}(a) &= 1, \\
\underline{\epsilon}(b) &= 0, \\
\underline{\epsilon}(c) &= 0, \\
\underline{\epsilon}(p) &= 1+q^{-2}
\end{aligned}
\tag{35}
$$

and the antipodes

$$
\begin{aligned}
\underline{S}(a) &= -q^{-2}a + p, \\
\underline{S}(b) &= -q^{-2}b,
\end{aligned}
\tag{36}
$$

$$\begin{aligned}\underline{S}(c) &= -q^{-2}c,\\ \underline{S}(p) &= p\end{aligned}$$

together with the braidings

$$\begin{aligned}\psi(a\underline{\otimes}a) &= a\underline{\otimes}a+(q^4-q^2)c\underline{\otimes}b,\\ \psi(a\underline{\otimes}b) &= b\underline{\otimes}a+(q^{-2}-q^2)a\underline{\otimes}b+(q^2-1)p\underline{\otimes}b,\\ \psi(a\underline{\otimes}c) &= c\underline{\otimes}a\\ \psi(b\underline{\otimes}b) &= q^{-2}b\underline{\otimes}b,\\ \psi(b\underline{\otimes}c) &= q^2c\underline{\otimes}b,\\ \psi(c\underline{\otimes}b) &= (q^2+1-q^{-2}-q^{-4})a\underline{\otimes}a+q^2b\underline{\otimes}c+(q^4-q^2+q^{-2}-1)c\underline{\otimes}b\\ &\quad +(q^{-2}-q^2)a\underline{\otimes}p+(q^{-2}-q^2)p\underline{\otimes}a+(q^2-1)p\underline{\otimes}p\end{aligned} \tag{37}$$

and their $\underline{*}$-involutions makes $B(R)$ a braided Hopf $\underline{*}$-algebra. Note that in the $q\to 1$ limit, not only the algebra becomes commutative but also the braided tensor product becomes the ordinary tensor product, and we obtain the bosonic statistics for both braided Hopf algebra solutions. When we express the solutions in terms of the original generators of the algebra, i.e., in terms of a,b,c,d by using (23) we see that that the first solution is completely equivalent to the braided Hopf algebra given in the literature which we give in (14)-(17). But the second solution with the coproducts

$$\begin{aligned}\underline{\Delta}(a) &= a\underline{\otimes}a+q^4c\underline{\otimes}b,\\ \underline{\Delta}(b) &= (1-q^2)a\underline{\otimes}b+b\underline{\otimes}a+q^2d\underline{\otimes}b,\\ \underline{\Delta}(c) &= (1-q^2)c\underline{\otimes}a+a\underline{\otimes}c+q^2c\underline{\otimes}d,\\ \underline{\Delta}(d) &= (q^2-1)a\underline{\otimes}a+(q^4-q^2)c\underline{\otimes}b+q^2b\underline{\otimes}c+(1-q^2)a\underline{\otimes}d+(1-q^2)d\underline{\otimes}a+q^2d\underline{\otimes}d\end{aligned} \tag{38}$$

counits

$$\underline{\epsilon}(a)=\underline{\epsilon}(d)=1,\quad \underline{\epsilon}(b)=\underline{\epsilon}(c)=0, \tag{39}$$

antipodes

$$\underline{S}(a)=d,\quad \underline{S}(b)=-q^{-2}b,\quad \underline{S}(c)=-q^{-2},\quad \underline{S}(d)=q^2d+(1-q^2)a \tag{40}$$

and braidings

$$\begin{aligned}\psi(a\underline{\otimes}a) &= a\underline{\otimes}a+(q^4-q^2)c\underline{\otimes}b,\\ \psi(a\underline{\otimes}b) &= b\underline{\otimes}a+(1-q^2)a\underline{\otimes}b+(q^2-1)d\underline{\otimes}b,\\ \psi(a\underline{\otimes}c) &= c\underline{\otimes}a,\end{aligned}$$

$$
\begin{aligned}
\psi(a\underline{\otimes}d) &= d\underline{\otimes}a+(1-q^2)c\underline{\otimes}b,\\
\psi(b\underline{\otimes}a) &= a\underline{\otimes}b,\\
\psi(b\underline{\otimes}b) &= q^{-2}b\underline{\otimes}b,\\
\psi(b\underline{\otimes}c) &= q^2c\underline{\otimes}b,\\
\psi(b\underline{\otimes}d) &= d\underline{\otimes}b, \qquad (41)\\
\psi(c\underline{\otimes}a) &= a\underline{\otimes}c+(1-q^2)c\underline{\otimes}a+(q^2-1)c\underline{\otimes}d,\\
\psi(c\underline{\otimes}b) &= (q^2-1)a\underline{\otimes}a+q^2b\underline{\otimes}c+(q^4-q^2+q^{-2}-1)c\underline{\otimes}b\\
&\quad +(1-q^2)a\underline{\otimes}d+(1-q^2)d\underline{\otimes}a+(q^2-1)d\underline{\otimes}d,\\
\psi(c\underline{\otimes}c) &= q^{-2}c\underline{\otimes}c,\\
\psi(c\underline{\otimes}d) &= d\underline{\otimes}c+(1-q^{-2})c\underline{\otimes}a+(q^{-2}-1)c\underline{\otimes}d,\\
\psi(d\underline{\otimes}a) &= a\underline{\otimes}d+(1-q^2)c\underline{\otimes}b,\\
\psi(d\underline{\otimes}b) &= b\underline{\otimes}d+(1-q^{-2})a\underline{\otimes}b+(q^{-2}-1)d\underline{\otimes}b,\\
\psi(d\underline{\otimes}c) &= c\underline{\otimes}d,\\
\psi(d\underline{\otimes}d) &= d\underline{\otimes}d+(1-q^{-2})c\underline{\otimes}b
\end{aligned}
$$

defines a different braided Hopf algebra.

4 A New Algebra and its Transmutation

We find that the nonbraided Hopf algebra generated by the generators a, b, c, d and 1 whose coalgebra is of the form (38)-(39) is a two parameter (q, r) deformed Hopf algebra. The algebra part is found to satisfy

$$
\begin{aligned}
a\cdot b &= rb\cdot a,\\
a\cdot c &= rc\cdot a,\\
b\cdot c &= c\cdot b, \qquad (42)\\
b\cdot d &= rd\cdot b+(q^{-2}-1)(r^2-1)b\cdot a\\
c\cdot d &= rd\cdot c+(q^{-2}-1)(r^2-1)c\cdot a,\\
a\cdot d-d\cdot a &= (r-r^{-1})q^2b\cdot c,\\
q^2d\cdot a+(1-q^2)a\cdot a-r^{-1}q^4c\cdot b &= 1.
\end{aligned}
$$

Finally the antipodes

$$S(a)=(1-q^2)a+q^2d,\ S(b)=-rb,\ S(c)=-r^{-1}c,\ S(d)=(2-q^2)a+(q^2-1)d \qquad (43)$$

completes Hopf algebra. For the $*$-algebra structure we find

$$a^* = (1-q^2)a + q^2 d, \quad b^* = -r^{-1}c, \quad c^* = -rb, \quad d^* = (2-q^2)a + (q^2-1)d. \tag{44}$$

The adjoint coaction of this Hopf algebra on itself calculated by using (4)

$$\begin{aligned}
\beta(a) &= a\otimes[(1-q^2)a\cdot a + q^2 d\cdot a - r^{-1}q^4(1-q^2)c\cdot b] + b\otimes[-r^{-1}q^4 c\cdot a] \\
&\quad + c\otimes[q^4(1-q^2)a\cdot b + q^6 d\cdot b] + d\otimes[-r^{-1}q^6 c\cdot b], \\
\beta(b) &= a\otimes[-q^2 a\cdot b] + b\otimes[a^2] + c\otimes[-q^4 rb\cdot b] + d\otimes[q^2 a\cdot b] \\
\beta(c) &= a\otimes[q^4 d\cdot c + q^2(1-q^2)a\cdot c] + c\otimes[(1-q^2)^2 a\cdot a + q^2(1-q^2)a\cdot d + q^2(1-q^2)d\cdot a + q^4 d\cdot d] \\
&\quad b\otimes[-q^4 r^{-1}c\cdot c] + d\otimes[q^2(q^2-1)r^{-1}c\cdot a + q^4 r^{-1}c\cdot d] \\
\beta(d) &= a\otimes[(q^6-q^4)r^{-1}c\cdot b - q^4 rc\cdot b] + c\otimes[(q^6-q^4-q^4r^2)d\cdot b + (q^4-q^2)(r^2-q^2+1)a\cdot b] \\
&\quad b\otimes[(q^2(1-q^2)r^{-1} + q^2 r)c\cdot a] + d\otimes[(1-q^2)a\cdot a + q^2 a\cdot d - (q^6-q^4)r^{-1}c\cdot b]
\end{aligned} \tag{45}$$

does not define an algebra homomorphism. However when the coacted copy satisfies the braided algebra (13), the transformations (45) define an algebra homomorphism for $r = q$. It can be shown that the braiding of the transformation in the braided tensor product space is equal to the transformation of the braiding on the same space, i.e.,

$$\psi(\beta(x\underline{\otimes}y)) = \beta(\psi(x\underline{\otimes}y)) \quad \forall x,y \in B(R) \tag{46}$$

where the braidings are given by (41). It can also be shown that the braided algebra is a comodule algebra under the adjoint coaction for both of the braided Hopf algebra solutions.

The transmuted multiplication $(\underline{\cdot})$ in terms of the multiplication $(\cdot)$ of the coacting nonbraided algebra

$$\begin{aligned}
a\underline{\cdot}a &= a\cdot a \\
a\underline{\cdot}b &= q^{-1}b\cdot a \\
a\underline{\cdot}c &= a\cdot c \\
a\underline{\cdot}d &= a\cdot d + (q-q^3)b\cdot c \\
b\underline{\cdot}a &= qb\cdot a \\
b\underline{\cdot}b &= q^{-1}b\cdot b \\
b\underline{\cdot}c &= qb\cdot c \\
b\underline{\cdot}d &= q^{-1}b\cdot d + q(1-q^{-2})^2 b\cdot a
\end{aligned} \tag{47}$$

$$
\begin{aligned}
c\,\underline{\cdot}\,a &= q^{-2}a\cdot c \\
c\,\underline{\cdot}\,b &= qc\cdot b+(q^{-2}-1)(d-a)\cdot a \\
c\,\underline{\cdot}\,c &= q^{-1}c\cdot c \\
c\,\underline{\cdot}\,d &= q^{-1}c\cdot d+(1-q^{-2})a\cdot c \\
d\,\underline{\cdot}\,a &= d\cdot a \\
d\,\underline{\cdot}\,b &= d\cdot b+(q^{-2}-q^{-4})a\cdot b \\
d\,\underline{\cdot}\,c &= d\cdot c \\
d\,\underline{\cdot}\,d &= d\cdot d+(q-q^{-1})b\cdot c
\end{aligned}
$$

completes the transmutation process.

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